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The Asymptotic Cost of Lagrange Interpolatory Side Conditions

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1. INTRODUCTION

In this paper we improve some results of Paszkowski [3, 4], and Johnson [2] concerning the cost of interpolatory side conditions.

Let T be compact Hausdorff, and let h, k belong to $C(T)$, the space of continuous real valued functions on T, and satisfy

$$h(t) < k(t), \qquad t \in T. \tag{1.1}$$

Define the set of functions

$$X = \{g \in C(T): h(t) \leq g(t) \leq k(t) \text{ for all } t \in T\}.$$

For convenience the case of no constraints will be denoted by $X = C(T)$.

Consider an increasing sequence of finite-dimensional linear subspaces $\{N_\nu\}_{\nu=1}^{\infty}$ of $C(T)$, whose union N is dense in $C(T)$, and the corresponding sequence of convex sets $M_\nu = N_\nu \cap X$ whose union M is clearly dense in X. Given a finite set $\{x_1^*,..., x_\gamma^*\}$ of elements of $C(T)^*$, the dual of $C(T)$, and $f \in C(T)$, define the set of functions

$$A = \{g \in C(T): x_i^*(g) = x_i^*(f), i = 1,..., \gamma\}.$$

For each $\nu = 1, 2, 3,...$ define $E_\nu = E_\nu(f)$ by

$$E_\nu(f) = \inf_{g \in M_\nu} \|f - g\| \tag{1.2}$$

where

$$\|f - g\| = \sup_{t \in T} |f(t) - g(t)|.$$

Similarly if $M_\nu \cap A$ is nonempty define

$$E_\nu(f, A) = \inf_{g \in (M_\nu \cap A)} \|f - g\|. \tag{1.3}$$

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Clearly $E_\nu(f) \leqslant E_\nu(f, A)$ whenever the right member exists. It is natural to ask if the ratio $E_\nu(f, A)/E_\nu(f)$ has upper bounds. Paszkowski [3, 4] first posed such questions, showing in [4];

THEOREM 1.1. *If $X = [a, b]$, $M_\nu = N_\nu$ is the space of algebraic polynomials of degree not exceeding ν for $\nu = 1, 2, 3,\ldots$, and $\{x_i^*\}_{i=1}^\gamma = \{f_{t_i}\}_{i=1}^\gamma$ are point evaluations*

$$x_i^*(f) = f(t_i), \qquad a \leqslant t_i \leqslant b, \qquad i = 1,\ldots, \gamma,$$

then there is a number ν_1, not depending on f, such that for all $f \in C[a, b]$ and $\nu \geqslant \nu_1$

$$E_\nu(f, A) \leqslant 2E_\nu(f).$$

More recently Johnson [2] has obtained theorems of a similar nature in a more general context. For the space $C(T)$ a general theorem of Johnson [2, Theorem 2.1] reduces to

THEOREM 1.2. *If $X = C(T)$, then given any $x_1^*,\ldots, x_\gamma^* \in X^*$, there exists a constant C and a positive integer ν_1, not depending on f, such that for every f in $C(T)$ and $\nu \geqslant \nu_1$, $E_\nu(f, A)$ is defined and*

$$E_\nu(f, A) \leqslant CE_\nu(f). \tag{1.4}$$

He also shows (Johnson [2, Theorem 3.5])

THEOREM 1.3. *Suppose $X = C(T)$ and $f \in C(T)$. Suppose $\{x_i^*\}_{i=1}^\gamma = \{f_{t_i}\}_{i=1}^\gamma$ are point evaluations on $C(T)$ such that*

$$|f(t_i)| < \|f\|, \qquad i = 1,\ldots, \gamma, \tag{1.5}$$

then there exist C and ν_1 such that for every $\nu \geqslant \nu_1$ there is an $m_\nu \in N_\nu$ for which

$$m_\nu(t_i) = f(t_i), \qquad i = 1,\ldots, \gamma, \tag{1.6}$$

$$\|m_\nu\| \leqslant \|f\|, \tag{1.7}$$

$$\|f - m_\nu\| \leqslant CE_\nu(f). \tag{1.8}$$

Using a result of Yamabe [5] we prove the following

THEOREM 1.4. *If $X = C(T)$, and $\{x_i^*\}_{i=1}^\gamma = \{f_{t_i}\}_{i=1}^\gamma$ are point evaluations on $C(T)$, then there exist ν_1 and a sequence $\{\delta_\nu\}_{\nu=\nu_1}^\infty$, not depending on f, such that for any $f \in C(T)$, $E_\nu(f, A)$ is defined for $\nu \geqslant \nu_1$ and*

$$E_\nu(f, A) \leqslant (2 + \delta_\nu)\, E_\nu(f), \qquad \nu \geqslant \nu_1,$$

where

$$\lim_{\nu\to\infty} \delta_\nu = 0.$$

THEOREM 1.5. *If* $\{x_i^*\}_{i=1}^{\gamma} = \{f_{t_i}\}_{i=1}^{\gamma}$ *are point evaluations on* $C(T)$ *and* $f \in X \backslash M$ *satisfies*

$$h(t_i) < f(t_i) < k(t_i), \qquad i = 1,..., \gamma,$$

then there exists a ν_1 *such that* $E_\nu(f, A)$ *is defined for* $\nu \geq \nu_1$ *and*

$$\limsup_{\nu\to\infty} (E_\nu(f, A)/E_\nu(f)) \leq 2.$$

By [2, Theorem 4.1], in which we may take the constant as 2, there follows

COROLLARY 1.6. *If* $X = C(T), f \in C(T)\backslash N$, *and* $f(t_i) < \|f\|$ *for* $i = 1,..., \gamma$ *then there exist a* ν_1 *and a sequence* $\{g_\nu\}_{\nu=\nu_1}^{\infty}$ *of* $g_\nu \in N_\nu$ *satisfying* $g_\nu(t_i) = f(t_i)$ *for* $i = 1,..., \gamma$, $\|g_\nu\| \leq \|f\|$ *and* $\limsup_{\nu\to\infty} (\|f - g_\nu\|/E_\nu) \leq 4$.

2. PROOFS OF THEOREMS

First we construct a function essential to both proofs.

By the Hausdorff property of T we can find disjoint open sets B_1 ,..., B_γ contaiing t_1 ,..., t_γ , respectively. $T\backslash B_j$ is closed $j = 1,..., \gamma$ and so also are the singletons $\{t_j\}$. Since compact Hausdorff implies normal the Urysohn theorem (see, e.g., Dugundji [1]) guarantees the existence of functions f_j , $j = 1,..., \gamma$ such that

$$f_j(t_j) = 1 \tag{2.1}$$

$$0 \leq f_j(t) \leq 1, \qquad t \in B_j, \tag{2.2}$$

$$f_j(t) = 0, \qquad t \in T\backslash B_j . \tag{2.3}$$

Proof of Theorem 1.4. Consider the following theorem of Yamabe [5].

THEOREM 2.1. *Let* M *be a dense convex subset of a real normed linear space* X *and let* $x_1^*,..., x_\gamma^* \in X^*$. *Then for each* $f \in X$ *and each* $\epsilon > 0$ *there exists a* $g \in M$ *such that* $\|f - g\| < \epsilon$ *and*

$$x_i^*(g) = x_i^*(f), \qquad i = 1,..., \gamma.$$

By this theorem there exist functions in N arbitrarily close to f_j which interpolate to f_j at the points t_i , $i = 1,..., \gamma$. Using also the finite dimensionality of the N_ν, there exists a ν_1 such that for $\nu \geqslant \nu_1$ there exist best approximations $q_{\nu j}$ from N_ν to f_j satisfying

$$q_{\nu j}(t_i) = f_j(t_i) = \delta_{ij}, \qquad i = 1,..., \gamma, \qquad j = 1,..., \gamma,$$

where δ_{ij} is the Kronecker delta, and also if

$$\delta_\nu = \max_{j=1,\ldots,\gamma} \| q_{\nu j} - f_j \|$$

then

$$\delta_\nu \to 0 \qquad \text{as } \nu \to \infty.$$

Let $q_{\nu 0}$ be a best approximation to f from N_ν. For $\nu \geqslant \nu_1$ define

$$q_\nu = q_{\nu 0} + \sum_{j=1}^{\gamma} (f(t_j) - q_{\nu 0}(t_j))\, q_{\nu j}.$$

Then q_ν interpolates f at the points t_j, $j = 1,..., \gamma$ and

$$\| q_\nu - f \| \leqslant \| q_{\nu 0} - f \| + \left\| \sum_{j=1}^{\gamma} (f(t_j) - q_{\nu 0}(t_j))\, q_{\nu j} \right\|.$$

Since the B_j are disjoint and

$$| q_{\nu j}(t) | \leqslant \begin{cases} 1 + \delta_\nu, & t \in B_j \\ \delta_\nu, & t \in T \backslash B_j \end{cases} \qquad j = 1,..., \gamma,$$

the second term on the right-hand side is bounded by

$$E_\nu(f)(1 + \gamma\delta_\nu).$$

The theorem now follows.

Proof of Theorem 1.5. In order to prove the theorem we need some lemmas.

LEMMA 2.2. *Let* X, f, *and* $f(t_i)$, $i = 1,..., \gamma$, *be as in the statement of Theorem* 1.5. *Then for each* $\epsilon > 0$ *there exists* $g \in M$ *satisfying*

$$g(t_i) = f(t_i), \qquad i = 1,..., \gamma, \tag{2.4}$$

$$\| g - f \| < \epsilon. \tag{2.5}$$

Proof. There exists $\epsilon_0 > 0$ such that if $\epsilon_0 \geqslant \epsilon > 0$ the function

$$f_\epsilon(t) = \begin{cases} k(t) - \epsilon & \text{if } f(t) > k(t) - \epsilon, \\ h(t) + \epsilon & \text{if } f(t) < h(t) + \epsilon, \\ f(t) & \text{otherwise,} \end{cases}$$

is continuous. Also there exists ϵ_1, $\epsilon_0 \geqslant \epsilon_1 > 0$ such that for $\epsilon_1 \geqslant \epsilon > 0$

$$f_\epsilon(t) = f(t_i), \qquad i = 1,..., \gamma.$$

For such an ϵ, by Yamabe's theorem (Theorem 2.1) applied to $C(T)$ and N there exists $g \in N$ such that

$$g(t_i) = f_\epsilon(t_i), i = 1,..., \gamma, \qquad \text{and} \qquad \| f_\epsilon - g \| < \epsilon.$$

Thus

$$g \in M, g(t_i) = f(t_i), i = 1,..., \gamma \qquad \text{and} \qquad \| f - g \| < 2\epsilon.$$

The result follows.

Given $f \in X$ satisfying the conditions of Theorem 1.5 define for each $j = 1,..., \gamma$, constants

$$a_j^+ = (k(t_j) - f(t_j))/2, \qquad a_j^- = (h(t_j) - f(t_j))/2,$$

and continuous functions

$$f_j^+(t) = \min[(a_j^+ f_j + f)(t), k(t)],$$
$$f_j^-(t) = \max[(a_j^- f_j + f)(t), h(t)].$$

By Lemma 2.2, and the finite dimensionality of the N_ν, there exists a ν_1 such that for $\nu \geqslant \nu_1$ there exist best approximations from M_ν, $p_{\nu j}^+$, $p_{\nu j}^-$ of f_j^+, f_j^-, respectively, satisfying

$$p_{\nu j}^+(t_i) = f_j^+(t_i), \qquad i = 1,..., \gamma,$$
$$p_{\nu j}^-(t_i) = f_j^-(t_i), \qquad i = 1,..., \gamma,$$

and the normalized maximum error in these approximations

$$\delta_\nu = \max_{j=1,...,\nu} \max(\| p_{\nu j}^+ - f_j^+ \|, \| p_{\nu j}^- - f_j^- \|) / \min_{i=1,...,\gamma} \min(| a_i^+ |, | a_i^- |)$$

converges to zero as ν goes to infinity. Let $p_{\nu 0}$ be a best approximation to f from M_ν. Define

$$\lambda_{\nu j}^+ = \max(0, (f(t_j) - p_{\nu 0}(t_j))/a_j^+),$$
$$\lambda_{\nu j}^- = \max(0, (f(t_j) - p_{\nu 0}(t_j))/a_j^-). \tag{2.6}$$

We note that $\lambda_{\nu j}^+$, $\lambda_{\nu j}^-$ are both nonnegative and at least one is zero. Define

$$p_{\nu j} = p_{\nu j}^+, \quad a_j = a_j^+, \qquad \text{if} \quad \lambda_{\nu j}^+ > 0,$$
$$p_{\nu j} = p_{\nu j}^-, \quad a_j = a_j^-, \qquad \text{if} \quad \lambda_{\nu j}^- > 0,$$
$$p_{\nu j} = p_{\nu j}^+, \quad a_j = a_j^+, \qquad \text{if} \quad \lambda_{\nu j}^+ = \lambda_{\nu j}^- = 0.$$

and

$$\lambda_{\nu j} = \lambda_{\nu j}^{+} + \lambda_{\nu j}^{-}, \qquad j = 1,\ldots, \gamma.$$

We choose $\nu_2 \geqslant \nu_1$ so large that $\lambda_{\nu j}$ is less than 1 for $j = 1,\ldots, \gamma$ and $\nu \geqslant \nu_2$. Then

LEMMA 2.3. *Let $\lambda_{\nu j}$, $p_{\nu j}$, ν_2 be defined as above. Then for all $\nu \geqslant \nu_2$ there exist*

$$\theta_i = \theta_i(\nu), \qquad i = 0,\ldots, \gamma$$

such that

$$\theta_0 > 0; \qquad \theta_i \geqslant 0, \qquad i = 1,\ldots, \gamma, \tag{2.7}$$

$$\sum_{i=0}^{\gamma} \theta_i = 1, \tag{2.8}$$

$$\left(\sum_{i=0}^{\gamma} \theta_i p_{\nu i}\right)(t_j) = f(t_j), \qquad j = 1,\ldots, \gamma, \tag{2.9}$$

$$\theta_i(\nu) \leqslant (1 + \epsilon_\nu)\, \lambda_{\nu i}, \qquad i = 1,\ldots, \gamma, \tag{2.10}$$

where

$$\epsilon_\nu \to 0 \qquad \text{as } \nu \to \infty. \tag{2.11}$$

Proof. The existence of $\{\theta_i\}_{i=1}^{\gamma}$ satisfying (2.7)–(2.9) can be established by induction.

Induction basis. Take $\theta_{00} = 1$.

Induction step. Given $\theta_{s0},\ldots, \theta_{ss} \geqslant 0$ such that

$$\theta_{s0} > 0; \qquad \theta_{si} \geqslant 0, \quad i = 1,\ldots, s; \quad \sum_{0}^{s} \theta_{si} = 1, \tag{2.12}$$

$$\sum_{i=0}^{s} \theta_{si} p_{\nu i}(t_j) = f(t_j), \qquad j = 1,\ldots, s, \tag{2.13}$$

for $s = \gamma_1$, $0 \leqslant \gamma_1 < \gamma$, we prove the existence of $\theta_{s+1,0},\ldots, \theta_{s+1,s+1}$ satisfying (2.12) and (2.13) for $s = \gamma_1 + 1$.

Take

$$p(\alpha) = (1 - \alpha)\left(\sum_{0}^{s} \theta_{si} p_{\nu i}(t_{s+1})\right) + \alpha p_{\nu,s+1}(t_{s+1}) - f(t_{s+1}).$$

If $p(0) = 0$, take $\theta_{s+1,i} = \theta_{si}$, $i \leqslant s$; $\theta_{s+1,s+1} = 0$. If $p(0) \neq 0$, then by the choice of $p_{\nu,s+1}$ and since $\lambda_{\nu j} < 1$, $p(0)$ lies on one side of 0 and $p(1)$ on the other. Hence by linearity of the function $p(\alpha)$ there is a unique α, $0 < \alpha < 1$ such that $p(\alpha) = 0$. Taking

$$\theta_{s+1,i} = \begin{cases} (1-\alpha)\,\theta_{si}, & i = 0,\dots, s, \\ \alpha & i = s+1, \end{cases}$$

we have the induction step.

It remains to show (2.10) and (2.11).

We note that on entering the inductive step we deal with a function

$$\sum_{i=0}^{s} \theta_{si}\, p_{\nu i}$$

whose value at t_{s+1} lies on the line segment joining $p_{\nu 0}(t_{s+1})$ and $f(t_{s+1})$. This shows that each $\theta_{s+1,s+1}$ is less than or equal to θ'_{s+1} where θ'_{s+1} is chosen so that $(1-\theta'_{s+1})\,p_{\nu 0} + \theta'_{s+1}\,p_{\nu,s+1}$ interpolates to f at t_{s+1}. Since the θ_{si}, $i = 0,\dots, s$, decrease towards the $\theta_i = \theta_i(\nu)$ it follows that

$$0 \leqslant \theta_i(\nu) \leqslant \theta_i', \qquad i = 1,\dots, \gamma. \tag{2.14}$$

Now if $\lambda_{\nu i}$ is zero then so is θ_i' and from (2.14), (2.10) holds. If $\lambda_{\nu i}$ is nonzero so is θ_i' and

$$\lambda_{\nu i} = \frac{(f - p_{\nu 0})(t_i)}{(p_{\nu i} - f)(t_i)}, \qquad \theta_i' = \frac{(f - p_{\nu 0})(t_i)}{(p_{\nu i} - p_{\nu 0})(t_i)}.$$

Thus

$$\frac{\theta_i'}{\lambda_{\nu i}} = \frac{(p_{\nu i} - f)(t_i)}{(p_{\nu i} - p_{\nu 0})(t_i)} \to 1 \qquad \text{as } \nu \to \infty \text{ through } \nu \text{ such that } \lambda_{\nu i} \neq 0.$$

This proves (2.10) and (2.11).

From the above lemma and the convexity of M_ν there exists for $\nu \geqslant \nu_2$

$$p_\nu^* = \sum_{i=0}^{\gamma} \theta_i(\nu)\, p_{\nu i}$$

in M_ν which interpolates to f at t_j, $j = 1,\dots, \gamma$.

Write $|p_\nu^*(t) - f(t)| \leqslant \sum_{i=0}^{\gamma} \theta_i(\nu)\,|p_{\nu i}(t) - f(t)|$.

Using the estimate of the last lemma, namely

$$0 \leqslant \theta_i(\nu) < (1+\epsilon_\nu)\,\lambda_{\nu i}$$

where $\epsilon_\nu \to 0$ as $\nu \to \infty$, the estimate

$$|p_{\nu i}(t) - f(t)| \leqslant |a_i|\,(|f_i(t)| + \delta_\nu)$$
$$\leqslant \begin{cases} |a_i|\,\delta_\nu & t \in T\setminus B_i, \\ |a_i|\,(1+\delta_\nu) & t \in B_i, \end{cases} \qquad i = 1,\dots, \gamma$$

and the estimates

$$\lambda_{\nu i} \mid a_i \mid \leqslant E_\nu(f), \qquad i = 1,..., \gamma,$$

we obtain

$$|p^*(t) - f(t)| \leqslant \begin{cases} E_\nu + (1 + \epsilon_\nu)(1 + \delta_\nu) E_\nu \\ \quad + (\gamma - 1)(1 + \epsilon_\nu) \delta_\nu E_\nu, & \text{if } t \in \bigcup_{i=1}^{\gamma} B_i \\ E_\nu + \gamma(1 + \epsilon_\nu) \delta_\nu E_\nu, & \text{if } t \in T/\left(\bigcup_{i=1}^{\gamma} B_i\right) \end{cases}$$

and writing $\delta_\nu' = \gamma(1 + \epsilon_\nu) \delta_\nu + \epsilon_\nu$

$$\| p^*(t) - f(t) \| \leqslant (2 + \delta_\nu') E_\nu .$$

This concludes the proof.

References

1. J. Dugundji, "Topology," Allyn & Bacon, Boston, Mass., 1966.
2. D. J. Johnson, Jackson type theorems for approximation with side conditions, *J. Approximation Theory* **12** (1974), 213–229.
3. S. Paszkowski, On the Weierstrass approximation theorem, *Colloq. Math.* **4** (1956), 206–210.
4. S. Paszkowski, On approximation with nodes, *Rozpr. Mat.* **14** (1957), 1–61.
5. H. Yamabe, On an extension of Hellys theorem, *Osaka J. Math.* **2** (1950), 15–17.